In the early days of modern mathematics, people were puzzled by equations like this one:

\[ x^2 + 1 = 0 \]

The equation looks simple enough, but in the sixteenth century people had no idea how to solve it. This is because to the common-sense mind the solution seems to be without meaning:

\[ x = \pm \sqrt{-1} \]

For this reason, mathematicians dubbed \( \sqrt{-1} \) an *imaginary number*. We abbreviate this by writing “\( i \)” in its place, that is:

\[ i = \sqrt{-1} \quad (1.1) \]
So we see that \( i^2 = -1 \), and we can solve equations like \( x^2 + 1 = 0 \). Note that electrical engineers use \( j = \sqrt{-1} \), but we will stick with the standard notation used in mathematics and physics.

**The Algebra of Complex Numbers**

More general complex numbers can be written down. In fact, using real numbers \( a \) and \( b \) we can form a complex number:

\[
c = a + ib
\]

We call \( a \) the real part of the complex number \( c \) and refer to \( b \) as the imaginary part of \( c \). The numbers \( a \) and \( b \) are ordinary real numbers. Now let \( c = a + ib \) and \( k = m + in \) be two complex numbers. Here \( m \) and \( n \) are also two arbitrary real numbers (not integers, we use \( m \) and \( n \) because I am running out of symbols to use). We can form the sum and difference of two complex numbers by adding (subtracting) their real and imaginary parts independently. That is:

\[
c + k = a + ib + m + in = (a + m) + i(b + n)
\]

\[
c - k = a + ib - (m + in) = (a - m) + i(b - n)
\]

To multiply two complex numbers, we just multiply out the real and imaginary parts term by term and use \( i^2 = -1 \), then group real and imaginary parts at the end:

\[
ck = (a + ib)(m + in) = am + ian + ibm + i^2bn
\]

\[
= am + ian + ibm - bn
\]

\[
= (am - bn) + i(an + bm)
\]

To divide two complex numbers and write the result in the form \( c = a + ib \), we’re going to need a new concept, called the complex conjugate. We form the complex conjugate of any complex number by letting \( i \rightarrow -i \). The complex conjugate is indicated by putting a bar on top of the number or variable. Again, let \( c = a + ib \). Then the complex conjugate is

\[
\bar{c} = a - ib
\]

It’s easy to see that if \( c \) is purely real, that is, \( c = a \), then the complex conjugate is \( \bar{c} = \bar{a} = a \). On the other hand, if \( c \) is purely imaginary, then \( c = ib \). This means that \( \bar{c} = ib = -ib = -c \). Taking the complex conjugate twice gives back the original number:

\[
\bar{\bar{c}} = \bar{a - ib} = a + ib = c
\]
Notice what happens when we multiply a complex number by its conjugate:

\[ c\overline{c} = (a + ib)(a - ib) = a^2 - iab + iba - i^2b^2 \]
\[ = a^2 - i^2b^2 = a^2 + b^2 \]

We call the quantity \( c\overline{c} \) the \textit{modulus} of the complex number \( c \) and write

\[ |c|^2 = c\overline{c} \quad (1.4) \]

Note that in physics, the complex conjugate is often denoted by an asterisk, that is, \( c^* \). The modulus of a complex number has geometrical significance. This is because we can view a complex number as a vector in the plane with components given by the real and imaginary parts. The length of the vector corresponds to the modulus. We will discuss this concept again later (see Fig. 1.1).

Now we can find the result of \( c/k \), provided that \( k \neq 0 \) of course. We have

\[
\frac{c}{k} = \frac{a + ib}{m + in}
\]
\[
= \frac{a + ib (m - in)}{m + in (m - in)}
\]
\[
= \frac{am + ibm - ian + bn}{m^2 + n^2}
\]
\[
= \frac{am + bn}{m^2 + n^2} + i \frac{bm - an}{m^2 + n^2}
\]

**Figure 1.1** The complex plane, showing \( z = x + iy \) and its complex conjugate as vectors.
We say that two complex numbers are equal if and only if their real and imaginary parts are equal. That is, \( c = a + ib \) and \( k = m + in \) are equal if and only if

\[
\begin{align*}
    a &= m \\
    b &= n \\
\Rightarrow c &= k
\end{align*}
\]

**Complex Variables**

In the early days, all of this probably seemed like a neat little trick that could be used to solve obscure equations, and not much more than that. But in reality it opened up a Pandora’s box of possibilities that is still being dealt with today. It turns out that an entire branch of analysis called complex analysis can be constructed, which really supersedes real analysis. Complex analysis has not only transformed the world of mathematics, but surprisingly, we find its application in many areas of physics and engineering. For example, we can use complex numbers to describe the behavior of the electromagnetic field. In atomic systems, which are described by quantum mechanics, complex numbers and complex functions play a central role, and actually appear to be a fundamental part of nature. Complex numbers are often hidden. For example, as we’ll see later, the trigonometric functions can be written down in surprising ways like:

\[
\begin{align*}
    \cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\
    \sin \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}
\end{align*}
\]

It appears that complex numbers are not so “imaginary” after all; rather they are used in a wide variety of engineering and science applications.

The first step in moving forward toward a calculus based on complex numbers is to abstract the notion of a complex number to a complex variable. This is the same as abstracting the notion of a real number to a variable like \( x \) that we can use to solve algebraic equations. We use \( z \) to represent a complex variable. Its real and imaginary parts are represented by the real variables \( x \) and \( y \), respectively. So we write

\[
z = x + iy	ag{1.5}
\]

The complex conjugate is then

\[
\bar{z} = x - iy
\]

A complex number and its conjugate have an interesting origin in the study of polynomials with real coefficients. Let \( p \) be a polynomial with real coefficients and suppose that a complex number \( z \) is a root of \( p \). Then it follows that the complex conjugate \( \bar{z} \) is a root of \( p \) also.
CHAPTER 1 Complex Numbers

The modulus of the complex variable $z$ is given by

\[ |z|^2 = x^2 + y^2 \quad \Rightarrow |z| = \sqrt{x^2 + y^2} \quad (1.6) \]

The same rules for addition, subtraction, multiplication, and division we illustrated with complex numbers apply to complex variables. So if $z = x + iy$ and $w = u + iv$ then

\[ zw = (x + iy)(u + iv) = (xu - yv) + i(yu + xv) \]

We can graph complex numbers in the $x$-$y$ plane, which we sometimes call the complex plane or the $z$ plane. The $y$ axis is the imaginary axis and the $x$ axis is the real axis. A complex number $z = x + iy$ can be depicted as a vector in the complex plane, with a length $r$ given by its modulus:

\[ r = |z| = \sqrt{x^2 + y^2} \quad (1.7) \]

We also keep track of the angle $\theta$ that this vector makes with the real axis. The complex conjugate is a vector reflected across the real axis. This is easy to understand since we form the conjugate by letting $y \rightarrow -y.$ These ideas are illustrated in Fig. 1.1.

**Rules for the Complex Conjugate**

Let $z = x + iy$ and $w = u + iv$ be two complex variables. Then

\[ \overline{z + w} = \overline{z} + \overline{w} \]

\[ \overline{zw} = \overline{z} \overline{w} \]

\[ \frac{\overline{z}}{\overline{w}} = \frac{\overline{z}}{\overline{w}} \quad (1.8) \]

These properties are easy to demonstrate. For example, we prove the first one:

\[ z + w = (x + iy) + (u + iv) = (x + u) + i(y + v) = (x + u) - i(y + v) = x - iy + u - iv = \overline{z} + \overline{w} \]
If \( z \neq 0 \), we can form the multiplicative inverse of \( z \) which we denote by \( z^{-1} \). The inverse has the property that

\[
zz^{-1} = 1 \quad (1.9)
\]

It is given by

\[
z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{1}{z} \quad (1.10)
\]

We can verify that this works explicitly in two ways:

\[
zz^{-1} = z \frac{\bar{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1
\]

\[
zz^{-1} = (x + iy) \frac{x - iy}{x^2 + y^2} = \frac{x^2 + y^2}{x^2 + y^2} = 1
\]

Notice that the inverse gives us a way to write the quotient of two complex numbers, allowing us to do division:

\[
\frac{z}{w} = \frac{z\bar{w}}{|w|^2} = \frac{z\bar{w}}{|w|^2}
\]

**EXAMPLE 1.1**

Find the complex conjugate, sum, product, and quotient of the complex numbers

\[ z = 2 - 3i \quad w = 1 + i \]

**SOLUTION**

To find the complex conjugate of each complex number we let \( i \rightarrow -i \). Hence

\[
\bar{z} = 2 - 3i = 2 + i \bar{3}
\]

\[
\bar{w} = 1 + i = 1 - i
\]
CHAPTER 1 Complex Numbers

The sum of the two complex numbers is formed by adding the real and imaginary parts, respectively:

\[ z + w = (2 - 3i) + (1 + i) = (2 + 1) + i(-3 + 1) = 3 - 2i \]

We can form the product as follows:

\[ zw = (2 - 3i)(1 + i) = 2 - 3i + 2i - 3i^2 = (2 + 3) + i(-3 + 2) = 5 - i \]

Finally, we use the complex conjugate of \( w \) to form the quotient:

\[ \frac{z}{w} = z \left( \frac{\bar{w}}{w} \right) = \frac{(2 - 3i)(1 - i)}{(1 + i)(1 - i)} = \frac{2 - 3i - 2i + 3i^2}{1 + i - i - i^2} = \frac{2 - 3 - 5i}{1 + 1} = \frac{-1 - 5i}{2} \]

**EXAMPLE 1.2**

Earlier, we said that if \( z = x + iy \), then \( x \) is the real part of \( z \) [denoted by writing \( x = \text{Re}(z) \) say] and that \( y \) is the imaginary part of \( z \) [\( y = \text{Im}(z) \)]. Derive expressions that allow us to define the real and imaginary parts of a complex number using only \( z \) and its complex conjugate.

**SOLUTION**

First let’s write down the complex variable and its complex conjugate:

\[ z = x + iy \quad \bar{z} = x - iy \]

Now we see that this is just simple algebra. We can eliminate \( y \) from both equations by adding them:

\[ z + \bar{z} = x + iy + x - iy = 2x \]

So, we find that the real part of \( z \) is given by

\[ \text{Re}(z) = x = \frac{z + \bar{z}}{2} \quad \text{(1.11)} \]
Now, let’s subtract the complex conjugate from \( z \) instead, which allows us to eliminate \( x \):

\[
z - \bar{z} = x + iy - (x - iy) = x + iy - x - iy = 2iy
\]

\[
\Rightarrow \text{Im}(z) = y = \frac{z - \bar{z}}{2i}
\]

(1.12)

**EXAMPLE 1.3**

Find \( z^2 \) if \( z = (2 + i)/(4i - (1 + 2i)) \).

**SOLUTION**

Note that when the modulus sign is not present, we square without computing the complex conjugate. That is \( |z|^2 = \bar{z}z \) but \( z^2 = z \cdot z \), which is a different quantity. So in this case we have

\[
z^2 = \left( \frac{2+i}{4i-(1+2i)} \right)^2
\]

\[
= \left( \frac{2+i}{4i-(1+2i)} \right) \left( \frac{2+i}{4i-(1+2i)} \right)
\]

\[
= \left( \frac{4+4i+i^2}{(1-2i)(-1+2i)} \right)
\]

\[
= \left( \frac{3+4i}{3-4i} \right)
\]

\[
= \left[ \frac{(3+4i)(-3+4i)}{(-3-4i)(-3+4i)} \right] \text{ (multiply and divide by complex conjugate of denominator)}
\]

\[
= \frac{-9 - 12i + 12i -16}{9 + 12i -12i + 16} = \frac{-25}{25} = -1
\]

**EXAMPLE 1.4**

Show that \( 1/i = -i \).

**SOLUTION**

This is easy, using the rule we’ve been applying for division. That is:

\[
\frac{z}{w} = \frac{z}{w} \left( \frac{\bar{w}}{\bar{w}} \right)
\]
CHAPTER 1  Complex Numbers

Hence

\[
\frac{1}{i} = \frac{1}{i} \left( \frac{-i}{-i} \right) = \frac{-i}{-i^2} = \frac{-i}{-(-1)} = -i
\]

EXAMPLE 1.5
Find \( z \) if \( z(7z + 14 - 5i) = 0 \).

SOLUTION
One obvious solution to the equation is \( z = 0 \). The other one is found to be

\[
7z + 14 - 5i = 0
\]

\[
\Rightarrow 7z = -14 + 5i
\]

or

\[
z = -2 + \frac{5}{7}i
\]

Pascal’s Triangle

Expansions of complex numbers can be written down immediately using Pascal’s triangle, which lists the coefficients in an expansion of the form \((x + y)^n\). We list the first five rows here:

\[
\begin{array}{c}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]  
(1.13)

The first row corresponds to \((x + y)^0\), the second row to \((x + y)^1\), and so on. For example, looking at the third row we have coefficients 1, 2, 1. This means that

\[
(x + y)^2 = x^2 + 2xy + y^2
\]

EXAMPLE 1.6
Write \((2 - i)^4\) in the standard form \(a + ib\).
SOLUTION
The coefficients for the fourth power are found in row five of Pascal’s triangle. In general:

\[(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\]

Hence

\[(2 - i)^4 = 2^4 + 4(2^3)(-i) + 6(2^2)(-i)^2 + 4(2)(-i)^3 + (-i)^4\]
\[= 16 - 32i + 24(-i)^2 + 8(-i)^3 + (-i)^4\]

Now let’s look at some of the terms involving powers of \(-i\) individually. First we have

\[(-i)^2 = (-1)^2i^2 = (+1)(-1) = -1\]

The last two terms are

\[(-i)^3 = (-1)^3i^3 = (-1)(i \cdot i^2) = (-1)(i)(-1) = +i\]
\[(-i)^4 = [(-i)^2]^2 = (-1)^2 = +1\]

Therefore we have

\[(2 - i)^4 = 16 - 32i + 24(-i)^2 + 8(-i)^3 + (-i)^4\]
\[= 16 - 32i - 24 + 8i + 1\]
\[= -7 - 24i\]

Axioms Satisfied by the Complex Number System

We have already seen some of the basics of how to handle complex numbers, like how to add or multiply them. Now we state the formal axioms of the complex number system which allow mathematicians to describe complex numbers as a
field. These axioms should be familiar since their general statement is similar to that used for the reals. We suppose that \( u, w, z \) are three complex numbers, that is, \( u, w, z \in \mathbb{C} \). Then these axioms follow:

\[
\begin{align*}
    z + w & \quad \text{and} \quad zw \in \mathbb{C} \quad \text{(closure law)} \\
    z + w &= w + z & \text{(commutative law of addition)} \\
    u + (w + z) &= (u + w) + z & \text{(associative law of addition)} \\
    zw &= wz & \text{(commutative law of multiplication)} \\
    u(wz) &= (uw)z & \text{(associative law of multiplication)} \\
    u(w + z) &= uw + uz & \text{(distributive law)}
\end{align*}
\]  

The identity with respect to addition is given by \( z = 0 + 0i \), which satisfies

\[
    z + 0 = 0 + z \quad (1.20)
\]

The identity with respect to multiplication is given by \( z = 1 + 0 = 1 \), which satisfies

\[
    z \cdot 1 = 1 \cdot z = z \quad (1.21)
\]

For any complex number \( z \) there exists an \textit{additive inverse}, which we denote by \(-z\) that satisfies

\[
    z + (-z) = (-z) + z = 0 \quad (1.22)
\]

There also exists a multiplicative inverse \( z^{-1} \), which we have seen satisfies

\[
    z z^{-1} = z^{-1} z = 1 \quad (1.23)
\]

A set that satisfies properties in Eqs. (1.14)–(1.23) is called a \textit{field}. The algebraic closure property in Eq. (1.14) illustrates that you can add two complex numbers together and you get another complex number (that is what we mean by \textit{closed}). The complex numbers are the smallest algebraically closed field that contains the reals as a subset.
Properties of the Modulus

We have already seen that the modulus or magnitude or absolute value of a complex number is defined by multiplying it by its complex conjugate and taking the positive square root. The absolute value operator satisfies several properties. Let $z_1, z_2, z_3, \ldots, z_n$ be complex numbers. Then

$$|z_1 z_2| = |z_1| |z_2|$$  \hspace{1cm} (1.24)

$$|z_1 z_2 z_3 \cdots z_n| = |z_1| |z_2| |z_3| \cdots |z_n|$$  \hspace{1cm} (1.25)

$$\frac{|z_1|}{|z_2|} = \frac{|z_1|}{|z_2|}$$  \hspace{1cm} (1.26)

A relationship called the triangle inequality deserves special attention:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$  \hspace{1cm} (1.27)

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$  \hspace{1cm} (1.28)

$$|z_1 + z_2| \geq |z_1| - |z_2|$$  \hspace{1cm} (1.29)

$$|z_1 - z_2| \geq |z_1| - |z_2|$$  \hspace{1cm} (1.30)

Also note that $w \overline{z} + \overline{w} = 2 \text{Re}(z \overline{w}) \leq 2|z||w|$.

The Polar Representation

In Fig. 1.1, we showed how a complex number can be represented by a vector in the $x$-$y$ plane. Using polar coordinates, we can develop an equivalent polar representation of a complex number. We say that $z = x + iy$ is the Cartesian representation of a complex number. To write down the polar representation, we begin with the definition of the polar coordinates $(r, \theta)$:

$$x = r \cos \theta \quad y = r \sin \theta$$  \hspace{1cm} (1.31)
We have already seen that when we represent a complex number as a vector in the plane the length of that vector is $r$. Hence, carrying forward with the vector analogy, the modulus of $z$ is given by

$$r = \sqrt{x^2 + y^2} = |x + iy|$$  \hspace{1cm} (1.32)

Using Eq. (1.31), we can write $z = x + iy$ as

$$z = x + iy = r \cos \theta + ir \sin \theta$$

$$= r(\cos \theta + i \sin \theta)$$  \hspace{1cm} (1.33)

Note that $r > 0$ and that we have $\tan \theta = y/x$ as a means to convert between polar and Cartesian representations.

**THE ARGUMENT OF Z**

The value of $\theta$ for a given complex number is called the argument of $z$ or $\arg z$. The principal value of $\arg z$ which is denoted by $\text{Arg } z$ is the value $-\pi < \Theta \leq \pi$. The following relationship holds:

$$\arg z = \text{Arg } z + 2n\pi \quad n = 0, \pm1, \pm2, \ldots$$  \hspace{1cm} (1.34)

The principal value can be specified to be between 0 and $2\pi$.

**EULER’S FORMULA**

*Euler’s formula* allows us to write the expression $\cos \theta + i \sin \theta$ in terms of a complex exponential. This is easy to see using a Taylor series expansion. First let’s write out a few terms in the well-known Taylor expansions of the trigonometric functions $\cos$ and $\sin$:

$$\cos \theta = 1 - \frac{1}{2} \theta^2 + \frac{1}{4!} \theta^4 - \frac{1}{6!} \theta^6 + \cdots$$  \hspace{1cm} (1.35)

$$\sin \theta = \theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \cdots$$  \hspace{1cm} (1.36)
Now, let’s look at $e^{i\theta}$. The power series expansion of this function is given by

$$e^{i\theta} = 1 + i\theta - \frac{1}{2!} \theta^2 + \frac{1}{3!} (i\theta)^3 + \frac{1}{4!} (i\theta)^4 + \frac{1}{5!} (i\theta)^5 + \cdots$$

$$= 1 + i\theta - \frac{1}{2} \theta^2 - i \frac{1}{3!} \theta^3 + \frac{1}{4!} \theta^4 + i \frac{1}{5!} \theta^5 + \cdots$$

(Now group terms—looking for sin and cosine)

$$= \left(1 - \frac{1}{2} \theta^2 + \frac{1}{4!} \theta^4 - \cdots \right) + \left(i\theta - i \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 + \cdots \right)$$

$$= \left(1 - \frac{1}{2} \theta^2 + \frac{1}{4!} \theta^4 - \cdots \right) + i \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 + \cdots \right)$$

$$= \cos \theta + i \sin \theta$$

So, we conclude that

$$e^{i\theta} = \cos \theta + i \sin \theta \quad (1.37)$$

$$e^{-i\theta} = \cos \theta - i \sin \theta \quad (1.38)$$

As noted in the introduction, these formulas can be inverted using algebra to obtain the following relationships:

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad (1.39)$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad (1.40)$$

These relationships allow us to write a complex number in complex exponential form or more commonly polar form. This is given by

$$z = re^{i\theta} \quad (1.41)$$

The polar form can be very useful for calculation, since exponentials are so simple to work with. For example, the product of two complex numbers $z = re^{i\theta}$ and $w = \rho e^{i\phi}$ is given by

$$zw = (re^{i\theta})(\rho e^{i\phi}) = r \rho e^{i(\theta + \phi)} \quad (1.42)$$
CHAPTER 1  Complex Numbers

Notice that moduli multiply and arguments add. Division is also very simple:

\[
\frac{z}{w} = \frac{re^{i\theta}}{\rho e^{i\phi}} = \frac{r}{\rho} e^{i(\theta - \phi)}
\]  

(1.43)

The reciprocal of a complex number takes on the relatively simple form:

\[
z = re^{i\theta} \implies z^{-1} = \frac{1}{r} e^{-i\theta}
\]  

(1.44)

Raising a complex number to a power is also easy:

\[
z^n = (re^{i\theta})^n = r^n e^{in\theta}
\]  

(1.45)

The complex conjugate is just

\[
\bar{z} = re^{-i\theta}
\]  

(1.46)

Euler’s formula can be used to derive some interesting expressions. For example, we can easily derive one of the most mysterious equations in all of mathematics:

\[
e^{i\pi} = \cos \pi + i \sin \pi
\]

\[
\Rightarrow e^{i\pi} + 1 = 0
\]  

(1.47)

DE MOIVRE’S THEOREM

Let \(z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)\) and \(z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)\). Using trigonometric identities and some algebra we can show that

\[
z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]
\]  

(1.48)

\[
z_1 / z_2 = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]
\]  

(1.49)

\[
z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n [\cos(\theta_1 + \theta_2 + \cdots + \theta_n) + i \sin(\theta_1 + \theta_2 + \cdots + \theta_n)]
\]  

(1.50)
De Moivre’s formula follows:

\[ z^n = [r(\cos \theta + i \sin \theta)]^n = r^n (\cos n\theta + i \sin n\theta) \quad (1.51) \]

### The \( n \)th Roots of Unity

Consider the equation

\[ z^n = 1 \]

where \( n \) is a positive integer. This innocuous looking equation actually has a bit of hidden data in it, this comes from the fact that

\[ (e^z)^n = e^z e^z \cdots e^z \]

The \( n \)th roots of unity are given by

\[ \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} = e^{\frac{2k\pi i}{n}} \quad k = 0, 1, 2, \ldots, n - 1 \quad (1.52) \]

If \( w = e^{\frac{2\pi i}{n}} \) then the \( n \) roots are \( 1, w, w^2, \ldots, w^{n-1} \).

**EXAMPLE 1.7**

Show that \( \cos z = \cos x \cosh y - i \sin x \sinh y \).

**SOLUTION**

This can be done using Euler’s formula:

\[
\cos(x + iy) = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{ixy} + e^{-ixy}}{2} = \frac{e^{ixy} + e^{-ixy} + e^{i(-y)} + e^{-i(-y)}}{4}
\]
CHAPTER 1 Complex Numbers

Now we can add and subtract some desired terms:

\[
\begin{align*}
\frac{e^{ix-y} + e^{-ix+y} + e^{ix+y} + e^{-ix-y}}{4} &= \frac{e^{ix+y} + e^{-ix+y} + e^{ix-y} + e^{-ix-y} + e^{ix+y} + e^{-ix+y} - e^{ix+y} - e^{-ix+y}}{4} \\
&= \frac{e^{ix+y} + e^{-ix+y} + e^{ix-y} + e^{-ix-y} - e^{ix+y} - e^{-ix+y} + e^{ix-y} + e^{-ix-y}}{4} \\
&= \left(\frac{e^{ix} + e^{-ix}}{2}\right)\left(\frac{e^{iy} + e^{-iy}}{2}\right) - \left(\frac{e^{ix} - e^{-ix}}{2}\right)\left(\frac{e^{iy} - e^{-iy}}{2}\right) \\
&= \left(\frac{e^{ix} + e^{-ix}}{2}\right)\left(\frac{e^{iy} + e^{-iy}}{2}\right) - i\left(\frac{e^{ix} - e^{-ix}}{2i}\right)\left(\frac{e^{iy} - e^{-iy}}{2}\right) \\
&= \cos x \cosh y - i \sin x \sinh y
\end{align*}
\]

EXAMPLE 1.8
Show that \( \sin^{-1} z = -i \ln(i z \pm \sqrt{1 - z^2}) \).

SOLUTION
We start with the relation

\[
\cos^2 \theta + \sin^2 \theta = 1
\]

This means that we can write

\[
\cos \theta = \pm \sqrt{1 - \sin^2 \theta}
\]

Now let \( \theta = \sin^{-1} z \). Then we have

\[
\cos(\sin^{-1} z) = \pm \sqrt{1 - \sin^2 (\sin^{-1} z)} = \pm \sqrt{1 - z^2}
\]

This is true because \( \sin(\sin^{-1} (\phi)) = \phi \). Now we turn to Euler’s formula:

\[
e^{i\phi} = \cos \theta + i \sin \theta
\]
Again, setting \( \theta = \sin^{-1} z \) we have
\[
e^{i\sin^{-1} z} = \cos(\sin^{-1} z) + i \sin(\sin^{-1} z)
\]
\[
= \pm \sqrt{1 - z^2} + iz
\]
Taking the natural logarithm of both sides, we obtain the desired result:
\[
i\sin^{-1} z = \ln(iz \pm \sqrt{1 - z^2})
\]
\[
\Rightarrow \sin^{-1} z = -i \ln(iz \pm \sqrt{1 - z^2})
\]

EXAMPLE 1.9
Show that \( e^{\ln z} = re^{i\theta} \).

SOLUTION
We use the fact that \( \theta = \theta + 2n\pi \) for \( n = 0, 1, 2, \ldots \) to get
\[
e^{\ln z} = e^{\ln(r e^{i\theta})}
\]
\[
= e^{\ln r + \ln e^{i\theta}}
\]
\[
= e^{\ln r + i\theta}
\]
\[
= e^{\ln r + (\theta + 2n\pi)}
\]
\[
= re^{i\theta} e^{i2n\pi}
\]
\[
= re^{i\theta} (\cos 2n\pi + i \sin 2n\pi) = re^{i\theta}
\]

EXAMPLE 1.10
Find the fourth roots of 2.

SOLUTION
We find the \( n \)th roots of a number \( a \) by writing \( r^n e^{i\theta} = a e^{i\theta} \) and equating moduli and arguments, and repeating the process by adding \( 2\pi \). This may not be clear, but we’ll show this with the current example. First we start out with
\[
(re^{i\theta})^4 = 2e^{i0}
\]
\[
\Rightarrow r = 2^{1/4} \quad \theta = 0
\]
This is the first of four roots. The second root is
\[
(re^{i\theta})^4 = r^4 e^{i4\theta} = 2e^{i2\pi}
\]
\[
\Rightarrow r = 2^{1/4} \quad \theta = \frac{\pi}{2}
\]
CHAPTER 1  Complex Numbers

So the second root is \( z = 2^{1/4} e^{i\pi/2} = 2^{1/4} [\cos(\pi/2) + i \sin(\pi/2)] = i2^{1/4} \). Next, we have

\[
(re^{i\theta})^4 = r^4 e^{i4\theta} = 2e^{i\pi}
\]

\[
\Rightarrow r = 2^{1/4} \quad \theta = \pi
\]

And the root is

\[
z = 2^{1/4} e^{i\pi} = 2^{1/4} (\cos \pi + i \sin \pi) = -2^{1/4}
\]

The fourth and final root is found using

\[
(re^{i\theta})^4 = r^4 e^{i4\theta} = 2e^{i6\pi}
\]

\[
\Rightarrow r = 2^{1/4} \quad \theta = \frac{3\pi}{2}
\]

In Cartesian form, the root is

\[
z = 2^{1/4} e^{i\pi} = 2^{1/4} \left( \cos \left( \frac{3\pi}{2} \right) + i \sin \left( \frac{3\pi}{2} \right) \right) = -i2^{1/4}
\]

Summary

The imaginary unit \( i = \sqrt{-1} \) can be used to solve equations like \( x^2 + 1 = 0 \). By denoting real and imaginary parts, we can construct complex numbers that we can add, subtract, multiply, and divide. Like the reals, the complex numbers form a field. These notions can be abstracted to complex variables, which can be written in Cartesian or polar form.

Quiz

1. What is the modulus of \( z = \frac{1-i}{4} \)?

2. Write \( z = \frac{2-4i}{3+2i-i^3} \) in standard form \( z = x + iy \).
3. Find the sum and product of $z = 2 + 3i, w = 3 - i$.

4. Write down the complex conjugates of $z = 2 + 3i, w = 3 - i$.

5. Find the principal argument of $\frac{i}{-2 - 2i}$.

6. Using De Moivre’s formula, what is $\sin 3\theta$?

7. Following the procedure outlined in Example 1.7, find an expression for $\sin(x + iy)$.

8. Express $\cos^{-1} z$ in terms of the natural logarithm.

9. Find all of the cube roots of $i$.

10. If $z = 16e^{i\pi}$ and $w = 2e^{i\pi/2}$, what is $\frac{z}{w}$?